数理解析研究所講究録 1334 巻 2003 年 158-174

On a location parameter family of distributions attaining the Bhattacharyya bound

筑波大・数学系 田中 秀和 (Hidekazu Tanaka)
Institute of Mathematics, University of Tsukuba

Summary

It is well known that, under appropriate regularity conditions, the variance of an unbiased estimator of a real-valued function of a real parameter can attain the Cramér-Rao lower bound only if the family of distributions is a one-parameter exponential family. But it seems that the necessary conditions about the probability distribution for which there exists an unbiased estimator whose variance coincides with the Bhattacharyya lower bound, are not completely known. The purpose of this paper is to specify a location parameter family of distributions attaining the general order Bhattacharyya bound. We also discuss the relation between the family of distributions attaining the Bhattacharyya bound and an exponential family of distributions involving a location parameter. Subsequently, these results are applied for a scale parameter family of distributions.

1. Introduction

For the lower bound of the variance of unbiased estimators, the Bhattacharyya inequality is known as a generalization of the Cramér-Rao one (Bhattacharyya (1946), see also Zacks (1971)). The Bhattacharyya inequality has been discussed by many authors from some point of view (Kakeshita (1962), Blight and Rao (1974), Mase (1977), Khan (1984)). It is well known that the family of distributions must be a one-parameter exponential family, if there exists an unbiased estimator whose variance coincides with the

---

AMS 2000 subject classification. Primary 62E10; secondary 62F10, 62E15.
Key words and phrases. Bhattacharyya inequality, location parameter, exponential family, differential equation.

Cramér-Rao lower bound (see Wijsman (1973)). Therefore, it seems to be natural to restrict the distributions for which there exists an unbiased estimator whose variance coincides with the Bhattacharyya bound to a distribution from an exponential family. It is shown by Fend (1959) that in some class the family of distributions attaining the Bhattacharyya bound is included in an exponential family of distributions. Further, a family of distributions attaining the Bhattacharyya bound is extended to the linear combination of distributions which belong to an exponential family (Tanaka and Akahira (2003), Tanaka (2003)). It is also shown in Tanaka and Akahira (2003) that the distribution which is not a linear combination of distributions belong to an exponential family can attain the Bhattacharyya bound. Hence, it seems to be unknown what is the family of distributions attaining the Bhattacharyya bound.

On the other hand, a necessary and sufficient condition for a location parameter family of distributions belong to an exponential one is derived (Ferguson (1962), see also Dynkin(1961)). Further, it is shown by Takeuchi (1973) that, among a location parameter family which has a one-dimensional minimal sufficient statistic, the distributions which have a uniformly minimum variance and unbiased (UMVU) estimator of a location parameter are limited to a normal distribution and an exp-gamma distribution in a regular case, and an exponential distribution in another case. (For the more general problem, see Bondesson (1975).) Consequently, this fact shows that a location parameter family of distributions attaining the Cramér-Rao bound is limited to the two distributions in the regular case.

In Section 3, we shall specify a location parameter family of distributions for which the variance of a UMVU estimator attains the general order Bhattacharyya bound under the suitable conditions. The restriction that the unknown parameter is a location parameter helps us to understand the structure of the family of distributions attaining the Bhattacharyya bound. In Section 4, we also discuss the relation between a location parameter family of distributions attaining the Bhattacharyya bound and an exponential family involving a location parameter. In section 5, the theory discussed in Section 3 is applied for a scale parameter family.

2. Bhattacharyya inequality

Let $(\mathcal{X}, \mathcal{B})$ be a sample space and suppose that a family of probability distributions $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ is dominated with respect to (w.r.t.) some σ-finite measure μ, where Θ is a parameter space which is an open interval of $\mathbb{R}^1$. Denote by $f(x, \theta) := dP_\theta/d\mu$ a probability density function (p.d.f.) w.r.t. μ. We consider an estimation problem of a

U-estimable function $g(\theta)$, i.e. the function $g(\theta)$ for which its unbiased estimator with a finite variance exists, based on a sample X.

We state an information inequality as the well-known Bhattacharyya one. Assume the following regularity conditions.

(A1) For μ-almost all x, $f(x,\theta)$ is k-times differentiable w.r.t. θ.

(A2) The integral $\int f(x,\theta)d\mu(x)$ can be i-times differentiated under the integral sign for each $i=1,\dots,k$.

(A3) $\int \left|((\partial^i/\partial\theta^i)f(x,\theta)(\partial^j/\partial\theta^j)f(x,\theta))/f(x,\theta)\right| d\mu(x) < \infty$ for all $\theta \in \Theta$ and for each $i,j=1,\dots,k$.

(A4) For μ-almost all x and for all $\theta \in \Theta$, $f(x,\theta) > 0$.

Let $I_k(\theta)$ be a $k \times k$ non-negative definite matrix with elements

$$E_\theta\left[\frac{(\partial^i/\partial\theta^i)f(X,\theta)}{f(X,\theta)}\cdot\frac{(\partial^j/\partial\theta^j)f(X,\theta)}{f(X,\theta)}\right],\ (i,j=1,\dots,k),$$

and $\boldsymbol{g}(\theta) := {}^t(g^{(1)}(\theta),\dots,g^{(k)}(\theta))$, where $g^{(i)}(\theta)$ is the i-th order derivative of $g(\theta)$ and ${}^tA$ denotes a transposition of a matrix A.

Theorem 1 (Bhattacharyya inequality). Suppose that the conditions (A1) to (A4) hold. Assume that $g(\theta)$ is a U-estimable function which is k-times differentiable over Θ. Let $\hat{g}(X)$ be an unbiased estimator of $g(\theta)$ with a finite variance, and assume that,

(A5) The integral $\int \hat{g}(x)f(x,\theta)d\mu(x)$ can be i-times differentiated under the integral sign for each $i=1,\dots,k$.

If $I_k(\theta)$ is non-singular over Θ, then

$$\text{(2.1)}\qquad \mathrm{Var}_\theta(\hat{g}(X)) \geq {}^t\boldsymbol{g}(\theta)I_k(\theta)^{-1}\boldsymbol{g}(\theta) =: B_k(\theta)$$

for all $\theta \in \Theta$. Here the equality holds in (2.1) if and only if

$$\text{(2.2)}\qquad \hat{g}(x) - g(\theta) = \sum_{i=1}^{k} a_{ki}(\theta)\frac{(\partial^i/\partial\theta^i)f(x,\theta)}{f(x,\theta)} \quad \mu\text{-}a.a.x$$

for all $\theta \in \Theta$, where $(a_{k1}(\theta),\dots,a_{kk}(\theta)) := {}^t\boldsymbol{g}(\theta)I_k(\theta)^{-1}$.

The proof is omitted, since it is given in Bhattacharyya (1946) and Zacks (1971). The lower bound $B_k(\theta)$ in (2.1) is called the k-th order Bhattacharyya (lower) bound. Note that $B_1(\theta)$ coincides with the Cramér-Rao lower bound. Further, we remark that the condition (A4) can be relaxed to the next condition to show (2.1) only.

(A4)' The support of $f(x,\theta)$, i.e. the set $\{x \in \mathcal{X} | f(x,\theta) > 0\}$ does not depend on θ.

But, here we assume the stronger condition (A4) in order to show (2.2).

Throughout the present paper, we treat only sufficiently smooth functions $f(x,\theta)$, $\hat{g}(x)$, $g(\theta)$, $a_{ki}(\theta)$ $(i = 1,\dots,k)$, since we investigate the distributions using differential equations. We shall say that the probability distribution (uniformly) attains the k-th order Bhattacharyya bound $B_k(\theta)$ if there exist an estimand $g(\theta)$ and its unbiased estimator $\hat{g}(x)$ such that $\mathrm{Var}_\theta(\hat{g}(X)) = B_k(\theta)$ for all $\theta \in \Theta$.

3. Location parameter family

In this section, we shall specify a location parameter family of distributions attaining the Bhattacharyya bound based on the equation (2.2). Suppose that $\mathcal{X} = \mathbb{R}^1$, $\Theta = \mathbb{R}^1$ and $f(x,\theta) = f(x-\theta)$, then (2.2) is reduced to

$$\hat{g}(x) = {}^t\tilde{a}_{k+1}(\theta)\tilde{h}_{k+1}(x-\theta) \tag{3.1}$$

for all x and all $\theta \in \mathbb{R}^1$, where

$$\tilde{a}_{k+1}(\theta) := {}^t(a_{k0}(\theta), a_{k1}(\theta), \dots, a_{kk}(\theta)), \qquad a_{k0}(\theta) := g(\theta),$$

$$\tilde{h}_{k+1}(u) := {}^t(h_0(u), h_1(u), \dots, h_k(u)), \qquad h_i(u) := (-1)^i f^{(i)}(u)/f(u).$$

Further, we denote

$$\hat{\boldsymbol{g}}_j(x) := {}^t\Big(\hat{g}(x), \hat{g}^{(1)}(x), \dots, \hat{g}^{(j-1)}(x)\Big), \qquad \boldsymbol{g}_j(\theta) := {}^t\Big(g(\theta), g^{(1)}(\theta), \dots, g^{(j-1)}(\theta)\Big),$$

$$\boldsymbol{h}_{ji}(u) := {}^t\Big(h_i(u), h_i^{(1)}(u), \dots, h_i^{(j-1)}(u)\Big), \quad \boldsymbol{a}_{ji}(\theta) := {}^t\Big(a_{ki}(\theta), a_{ki}^{(1)}(\theta), \dots, a_{ki}^{(j-1)}(\theta)\Big)$$

for each i, j. Although we are concerned with the p.d.f. $f(u)$ which satisfies the condition (3.1) in addition to the conditions (A1) to (A5), we focus only (3.1) for the present. Put

$$\mathcal{A}_k := \left\{\tilde{a}_{k+1}(\cdot) \mid \exists\theta_0 \in \mathbb{R}^1 \text{ s.t. } \left|(\boldsymbol{a}_{k+1,0}(\theta), \dots, \boldsymbol{a}_{k+1,k}(\theta))\right| \neq 0\right\},$$

$$\mathcal{B}_k := \left\{f(\cdot) | \exists\hat{g}(\cdot) \text{ and } \exists\tilde{a}_{k+1}(\cdot) \in \mathcal{A}_k \text{ s.t. (3.1) is satisfied for all } x, \theta \in \mathbb{R}^1\right\}.$$

When $\tilde{a}_{k+1} \in \mathcal{A}_k$, let $\theta_0 \in \mathbb{R}^1$ be the point such that $|(\boldsymbol{a}_{k+1,0}(\theta_0), \dots, \boldsymbol{a}_{k+1,k}(\theta_0))| \neq 0$. We consider the $(k+1)$-th order differential equations

$$(3.2)_i \qquad \left|\left(\boldsymbol{h}^{(1)}_{k+1,i}(u), \boldsymbol{a}_{k+1,1}(\theta_0), \ldots, \boldsymbol{a}_{k+1,k}(\theta_0)\right)\right| = 0,$$

and put

$$\mathcal{F}_{ki} := \left\{f(\cdot)\,\middle|\,(3.2)_i \text{ is satisfied for all } u \in \mathbb{R}^1\right\}$$

for $i = 1, \ldots, k$.

Lemma 3.1. It holds that $\mathcal{B}_k \subset \cap_{i=1}^k \mathcal{F}_{ki}$.
Proof. Let $f \in \mathcal{B}_k$, then by induction, we get

$$\hat{g}^{(j)}(x) = {}^t\tilde{a}^{(j)}_{k+1}(\theta)\tilde{h}_{k+1}(x-\theta),$$

which implies

$$(3.3) \qquad \hat{\boldsymbol{g}}^{(j)}_{k+1}(x) = (\boldsymbol{a}_{k+1,0}(\theta), \ldots, \boldsymbol{a}_{k+1,k}(\theta))\, \tilde{h}^{(j)}_{k+1}(x-\theta),$$

for each j. This leads to

$$(3.4) \qquad \left|\left(\hat{\boldsymbol{g}}^{(1)}_{k+1}(x), \boldsymbol{a}_{k+1,1}(\theta_0), \ldots, \boldsymbol{a}_{k+1,k}(\theta_0)\right)\right| = 0,$$

since $h_0(x-\theta) = 1$. Further, by arranging the equalities (3.3) for $j = 1, \ldots, k+1$, we have

$$\left(\hat{\boldsymbol{g}}^{(1)}_{k+1}(x), \ldots, \hat{\boldsymbol{g}}^{(k+1)}_{k+1}(x)\right) = (\boldsymbol{a}_{k+1,0}(\theta_0), \ldots, \boldsymbol{a}_{k+1,k}(\theta_0)) \left(\tilde{h}^{(1)}_{k+1}(x-\theta_0), \ldots, \tilde{h}^{(k+1)}_{k+1}(x-\theta_0)\right)$$

By considering the transposed matrix, it can be rewritten as

$$\left(\hat{\boldsymbol{g}}^{(\cdot)}_{k+1}(x), \ldots, \hat{\boldsymbol{g}}^{(k+1)}_{k+1}(x)\right) = \left(\boldsymbol{h}^{(1)}_{k+1,0}(x-\theta_0), \ldots, \boldsymbol{h}^{(1)}_{k+1,k}(x-\theta_0)\right) \left(\tilde{a}_{k+1}(\theta), \ldots, \tilde{a}^{(k)}_{k+1}(\theta)\right),$$

which implies that

$$\left|\left(\boldsymbol{h}^{(1)}_{k+1,i}(u), \boldsymbol{a}_{k+1,1}(\theta_0), \ldots, \boldsymbol{a}_{k+1,k}(\theta_0)\right)\right| = 0$$

for all $u \in \mathbb{R}^1$ and for each $i = 1, \ldots, k$, by (3.4) and a multilinerity of a determinant. □

Lemma 3.2. For $m = 1, \ldots, k$, $h_m(u)$ can be represented as

$$(3.5) \qquad h_m(u) = h_1^m(u) + P_{m-1}(\boldsymbol{h}_{m1}(u)),$$

where $P_{m-1}(\boldsymbol{h}_{m1})$ is a polynomial in $\boldsymbol{h}_{m1}$ of degree $m-1$ at most.

Proof. Obviously, (3.5) holds for $m = 1$. By induction, suppose that (3.5) is true for $m = 1, \dots, m_0$. Then we have

$$\begin{aligned} h_{m_0+1}(u) &= h_1(u)h_{m_0}(u) - h_{m_0}^{(1)}(u) \\ &= h_1(u)\left\{h_1^{m_0}(u) + P_{m_0-1}(\boldsymbol{h}_{m_0,1}(u))\right\} - m_0 h_1^{m_0-1}(u)h_1^{(1)}(u) \\ &\quad - \sum_{i=0}^{m_0-1} h_1^{(i+1)}(u)\left[\frac{\partial}{\partial h_1^{(i)}}P_{m_0-1}(\boldsymbol{h}_{m_0,1})\right]_{\boldsymbol{h}_{m_0,1}=\boldsymbol{h}_{m_0,1}(u)} \\ &= h_1^{m_0+1}(u) + P_{m_0}\left(\boldsymbol{h}_{m_0+1,1}(u)\right). \end{aligned}$$

This completes the proof. □

Now, since we are concerned with the solution of the $(k+1)$-th order differential equation

$$\left|\left(\boldsymbol{h}_{k+1}^{(1)}(u), \boldsymbol{a}_{k+1,1}(\theta_0), \dots, \boldsymbol{a}_{k+1,k}(\theta_0)\right)\right| = 0, \tag{3.6}$$

where $\boldsymbol{h}_{k+1}(u) := {}^t(h(u), h^{(1)}(u), \dots, h^{(k)}(u))$, we consider the $(k+1)$-th order algebraic equation

$$|(\boldsymbol{z}_{k+1}, \boldsymbol{a}_{k+1,1}(\theta_0), \dots, \boldsymbol{a}_{k+1,k}(\theta_0))| = 0, \tag{3.7}$$

where $\boldsymbol{z}_{k+1} := {}^t(z, z^2, \dots, z^{k+1})$. Let S be a set of solutions of (3.6), and we shall call $h(u)$ the highest term of $h_m(u)$ if $|h_m(u)/h(u)| \to c$ $(0 < c < \infty)$ as $u \to \infty$.

Lemma 3.3. If $f \in \mathcal{B}_k$, then h_1 is limited to the case when

$$h_1(u) = \begin{cases} H_1 + H_2 u & \text{if the solutions of (3.7) are only 0,} \\ H_1 + H_2 e^{z_0 u} & \text{if the solutions of (3.7) are } 0, z_0, z_0^2, \dots, z_0^k, \\ H_1 & \text{otherwise,} \end{cases}$$

where H_1 and H_2 are arbitrary complex constants and z_0 is arbitrary non-zero complex constant.

Proof. (I) When the solutions of (3.7) are only 0, we see $S = \mathrm{Span}[u^{\delta-1} | \delta = 1, \dots, k+1]$, which denotes a linear space spanned by $u^{\delta-1}$ for $\delta = 1, \dots, k+1$. Further, $h_1(u)$ is represented as

$$h_1(u) = \sum_{\delta=1}^{l} H_\delta u^{\delta-1}$$

for some l $(1 \le l \le k+1)$ by Lemma 3.1, where H_δ $(\delta = 1, \dots, l)$ are arbitrary complex constants. Then by Lemma 3.2, we see that the highest term of $h_k(u)$ is that of $h_1^k(u)$, that is, $H_l^k u^{k(l-1)}$. On the other hand, the highest term of the elements of S is u^k at most, which leads to $H_l = 0$ for each l $(3 \le l \le k+1)$.

(II) When the solutions of (3.7) are not only 0, we classify the solutions of (3.7) according to the absolute into $\sum_{\alpha=1}^{\kappa} \xi_\alpha + 1$ groups like

$$\begin{cases} 0, & & & & & \\ z_{11}, & z_{12}, & \cdots & z_{1\xi_1}, & & \\ z_{21}, & z_{22}, & \cdots & \cdots & \cdots & z_{2\xi_2}, \\ \vdots & \vdots & & & & \\ z_{\kappa 1}, & z_{\kappa 2}, & \cdots & \cdots & z_{\kappa\xi_\kappa}, & \end{cases}$$

where $|z_{\alpha\beta}| = |z_{\alpha\beta'}|$ for $\alpha = 1, \ldots, \kappa$ and $\beta, \beta' = 1, \ldots, \xi_\alpha, 0 < |z_{1\cdot}| < |z_{2\cdot}| < \cdots < |z_{\kappa\cdot}|$ and let w_0 and $w_{\alpha\beta}$ be the algebraic multiplicities of 0 and $z_{\alpha\beta}$, respectively. Then

$$S = \operatorname{Span}\left[u^{\delta-1}, u^{\gamma-1} e^{z_{\alpha\beta} u} \,\middle|\, \delta = 1, \ldots, w_0; \alpha = 1, \ldots, \kappa; \beta = 1, \ldots, \xi_\alpha; \gamma = 1, \ldots, w_{\alpha\beta} \right]$$

and $h_1(u)$ can be represented as

$$h_1(u) = \sum_{\delta=1}^{w_0} H_\delta u^{\delta-1} + \sum_{\alpha=1}^{l} \sum_{\beta=1}^{\xi_\alpha} \sum_{\gamma=1}^{w_{\alpha\beta}} H_{\alpha\beta\gamma} u^{\gamma-1} e^{z_{\alpha\beta} u}$$

for some l $(1 \le l \le \kappa)$, where $H_{\alpha\beta\gamma}$ and H_δ are arbitrary complex constants and

$$\sum_{\alpha=1}^{\kappa} \sum_{\beta=1}^{\xi_\alpha} w_{\alpha\beta} + w_0 = k + 1. \tag{3.8}$$

Let $Q_{w_{l\beta}-1,m}(u) := \sum_{\gamma=1}^{w_{l\beta}} H_{l\beta\gamma}^m u^{m(\gamma-1)}$. Then we note that $Q_{w_{l\beta}-1,1}(u) = 0$ for all u if and only if $Q_{w_{l\beta}-1,m}(u) = 0$ for all u and each m. Therefore, the terms $Q_{w_{l\beta}-1,m}(u) e^{m z_{l\beta} u}$ in $h_m(u)$ vanish if and only if $Q_{w_{l\beta}-1,1}(u) = 0$ for all u. Put $Z := \{z_{\alpha\beta} \,|\, \alpha = 1, \ldots, \kappa; \beta = 1, \ldots, \xi_\alpha\}$. Then we see from (3.8) that

$$k > \#\{z \in Z \,|\, |z_{l\cdot}| \le |z|\} \tag{3.9}$$

for $l \ge 2$, and

$$k \ge \#\{z \in Z \,|\, |z_{1\cdot}| \le |z|\}. \tag{3.10}$$

Now, we suppose that for $l \ge 2$, there exist β $(1 \le \beta \le \xi_l)$ and u such that $Q_{w_{l\beta}-1,1}(u) \ne 0$. Since $h_m(u) \in S$ for $m = 1, \ldots, k$, it must satisfy that $e^{m z_{l\beta} u} \in S$ for $m = 1, \ldots, k$. But this contradicts (3.9). Hence we get $Q_{w_{l\beta}-1,1}(u) = 0$ for all u and each β.

Next, we suppose that there exist β $(1 \le \beta \le \xi_1)$ and u such that $Q_{w_{1\beta}-1,1}(u) \ne 0$. Then by the same way as $l \ge 2$, it follows that $e^{m z_{1\beta} u} \in S$ for $m = 1, \ldots, k$. Further, by

(3.10), we see that for only $\beta = 1$, there is a possibility that $Q_{w_{1\beta}-1,1}(u) \neq 0$ for some u. Hence $h_1(u)$ is deduced to

$$h_1(u) = \sum_{\delta=1}^{w_0} H_\delta u^{\delta-1} + Q_{w_{11}-1,1}(u) e^{z_{11}u}.$$

Here we consider the next two cases.
(i) The case when $Q_{w_{11}-1,1}(u) \neq 0$ for some u. Since $h_m(u) \in S$ for $m = 1, \dots, k$, it follows that $S = \mathrm{Span}\left[1, e^{z_{11}u}, \dots, e^{kz_{11}u}\right]$, thus

$$h_1(u) = H_1 + H_{111} e^{z_{11}u}.$$

(ii) The case when $Q_{w_{11}-1,1}(u) = 0$ for all u. Since $h_1(u)$ can be represented as

$$h_1(u) = \sum_{\delta=1}^{l} H_\delta u^{\delta-1}$$

for some l $(1 \leq l \leq w_0)$, it follows that the highest term of $h_{w_0}(u)$ is $H_l^{w_0} u^{w_0(l-1)}$. So we get $H_l = 0$ for $l \geq 2$, hence $h_1(u) = H_1$. $\square$

Lemma 3.3 is essential to get the next main theorem.

Theorem 2. A location parameter family of distributions attaining the Bhattacharyya bound consists of a normal distribution and an exp-gamma distribution.
Proof. By Lemma 3.3, we get

$$f(u) = \begin{cases} \exp\{-(H_0 + H_1 u + H_2 u^2/2)\} & \text{if the solutions of (3.7) are only } 0, \\ \exp\{-(H_0 + H_1 u + H_2 e^{z_0 u}/z_0)\} & \text{if the solutions of (3.7) are } 0, z_0, z_0^2, \dots, z_0^k, \\ \exp\{-(H_0 + H_1 u)\} & \text{otherwise,} \end{cases}$$

whe.e H_0, H_1 and H_2 are arbitrary complex constants and z_0 is arbitrary non-zero complex constant. We consider whether the distribution uniformly attains the Bhattacharyya bound for each cases.
(I) First case (normal). It follows that $f(u) \in \mathbb{R}^1$ if and only if

$$\exp\left\{-\left(H_0 + H_1 u + \frac{H_2}{2}u^2\right)\right\} = \exp\left\{-\left(\overline{H_0} + \overline{H_1} u + \frac{\overline{H_2}}{2}u^2\right)\right\},$$

where $\overline{z}$ denotes the conjugate complex of z. Further, from the condition that $f(u) > 0$ for all $u \in \mathbb{R}^1$, we get $e^{-H_0} > 0$ and $H_1, H_2 \in \mathbb{R}^1$. It is well known that this function is a p.d.f. over $(\mathbb{R}^1, \mathcal{B})$ if and only if $f(u)$ can be represented as

$$f(u) = \frac{1}{\sqrt{2\pi}b} \exp\left\{-\frac{(u-a)^2}{2b^2}\right\} \quad (u \in \mathbb{R}^1; a \in \mathbb{R}^1, b > 0).$$

By (3.4) we see that $\hat{g}(x)$ is a polynomial in x of degree k at most, and $g(\theta)$ is also so, since $g(\theta)$ is an expected value of $\hat{g}(X)$. So it can be rewritten as

$$\hat{g}(x) = \sum_{j=0}^{k} \hat{G}_j x^j \quad \text{and} \quad g(\theta) = \sum_{j=0}^{k} G_j \theta^j$$

for some constants $\hat{G}_j$ and G_j, where G_j ($j = 0, 1, \ldots, k$) may depend on $\hat{G}_j$ ($j = 0, 1, \ldots, k$). Further, we see that $h_m(u)$ is a (Hermite) polynomial in $u - a$ of degree m by Lemma 3.2. Therefore, there exists a non-singular matrix U_{k+1} such that

$${}^t\!\left(1, u-a, \ldots, (u-a)^k\right) = U_{k+1}\tilde{h}_{k+1}(u).$$

By using this matrix, we get the relation

$$\hat{g}(x) = (C_0(\theta), C_1(\theta), \ldots, C_k(\theta))\, U_{k+1}\tilde{h}_{k+1}(u),$$

where $C_m(\theta) := \sum_{j=m}^{k} \hat{G}_j \binom{j}{m} (a+\theta)^{j-m}$ for $m = 0, 1, \ldots, k$, which shows that (3.1) is satisfied.

(II) Second case (exp-gamma). In a similar way to the case (I), we get

$$\exp\left\{-\left(H_0 + H_1 u + \frac{H_2}{z_0} e^{z_0 u}\right)\right\} = \exp\left\{-\left(\overline{H_0} + \overline{H_1} u + \frac{\overline{H_2}}{\overline{z_0}} e^{\overline{z_0} u}\right)\right\}$$

for all $u \in \mathbb{R}^1$. When $z_0 \neq \overline{z_0}$, we see that $f(u) = \exp\{-(H_0 + H_1 u)\}$ ($u \in \mathbb{R}^1; e^{-H_0} > 0, H_1 \in \mathbb{R}^1$). In this case $f(u)$ is not a p.d.f. over $(\mathbb{R}^1, \mathcal{B})$, since the definite integral of $f(u)$ over $\mathbb{R}^1$ diverges for any H_0 and H_1. When $z_0 = \overline{z_0}$, we see that

$$f(u) = \exp\left\{-\left(H_0 + H_1 u + \frac{H_2}{z_0} e^{z_0 u}\right)\right\} \quad \left(u \in \mathbb{R}^1; e^{-H_0} > 0, H_1, H_2 \in \mathbb{R}^1, z_0 \in \mathbb{R}^1 \setminus \{0\}\right).$$

It can be easily shown that this function is a p.d.f. over $(\mathbb{R}^1, \mathcal{B})$ if and only if $f(u)$ can be represented as

$$f(u) = \frac{|b|}{\Gamma(a)c^a} \exp\left\{-\frac{e^{bu}}{c} + abu\right\} \quad (u \in \mathbb{R}^1; a, c > 0, b \neq 0).$$

Further, $\hat{g}(x)$ and $g(\theta)$ can be represented as

$$\hat{g}(x) = \sum_{j=0}^{k} \hat{G}_j e^{jbx} \quad \text{and} \quad g(\theta) = \sum_{j=0}^{k} G_j e^{jb\theta}$$

for some constants $\hat{G}_j$ and G_j. Since $h_m(u)$ is a polynomial in e^{bu} of degree m, there exists a non-singular matrix U_{k+1} such that

$${}^t\!\left(1, e^{bu}, \ldots, e^{kbu}\right) = U_{k+1}\tilde{h}_{k+1}(u).$$

Then we obtain

$$\hat{g}(x) = \left(\hat{G}_0, \hat{G}_1 e^{b\theta}, \ldots, \hat{G}_k e^{kb\theta}\right) U_{k+1} \tilde{h}_{k+1}(u),$$

which shows that (3.1) is satisfied.

(III) Final case (exponential). In this case, it is impossible to consider the Bhattacharyya inequality since $f(u)$ is not a p.d.f. over $(\mathbb{R}^1, \mathcal{X})$. $\square$

Remark 3.1. A location parameter family of distributions attaining the Bhattacharyya bound coincides with that of distributions with a UMVU estimator of a location parameter, under the condition that the dimension of the minimal sufficient statistic is one (Takeuchi (1973)). But, if this condition is violated, then the former family is strictly narrower than the latter one. For example, suppose that a random variable X is distributed according to the p.d.f. $f(x-\theta) = C\exp\left\{-(x-\theta)^4\right\}$ $(x \in \mathbb{R}^1; \theta \in \mathbb{R}^1)$, where C is the normalizing constant (Bondesson (1975)). Let $g(\theta) := \theta$ be an estimand. Then the UMVU estimator of $g(\theta)$ is X, but it can be easily shown that the variance of X does not attain the k-th order Bhattacharyya bound for any k. (See also Remark 4.2 in the next section.)

4. The relation between a family of distributions attaining the Bhattacharyya bound and an exponential family

An exponential family involving one location parameter has derived by Dynkin (1961) and Ferguson (1962). In this section, we shall specify this family by a differential equation approach which helps us to understand the relation between a family of distributions attaining the Bhattacharyya bound and an exponential family. First, we suppose that the p.d.f. $f(x-\theta)$ w.r.t. μ is given by

$$f(x-\theta) = \exp\left\{{}^t\boldsymbol{t}_\kappa(x)\boldsymbol{s}_\kappa(\theta) + s_0(\theta) + t_0(x)\right\}, \tag{4.1}$$

for all $x, \theta \in \mathbb{R}^1$, where $\boldsymbol{s}_\kappa(\theta) := {}^t(s_1(\theta), \ldots, s_\kappa(\theta))$, $\boldsymbol{t}_\kappa(x) := {}^t(t_1(x), \ldots, t_\kappa(x))$ and the dimension of $\mathrm{Span}[1, s_l(\theta) | l = 1, \ldots, \kappa]$ is $\kappa + 1$. Let $\kappa + r + 1$ be the dimension of $\mathrm{Span}[1, s_l(\theta), s_l^{(1)}(\theta) | l = 1, \ldots, \kappa]$. Put

$$\mathcal{E}_\kappa := \left\{f(\cdot) \,|\, \exists(\boldsymbol{s}_\kappa(\cdot), \boldsymbol{t}_\kappa(\cdot)) \text{ s.t. (4.1) is satisfied for all } x, \theta \in \mathbb{R}^1\right\}.$$

Further, we consider the differential equation

$$\left|\left(\boldsymbol{h}^{(1)}_{k+1,i}(u), \boldsymbol{a}^0_{k+1,1}, \ldots, \boldsymbol{a}^0_{k+1,k}\right)\right| = 0, \tag{4.2$_i$}$$

and put

$$\mathcal{D}_{ki} := \left\{f(\cdot) \,\middle|\, \exists(\boldsymbol{a}^0_{k+1,1}, \ldots, \boldsymbol{a}^0_{k+1,k}) \text{ s.t. (4.2)}_i \text{ is satisfied for all } u \in \mathbb{R}^1\right\}.$$

Then it clearly holds that $\mathcal{F}_{ki} \subset \mathcal{D}_{ki}$ for each $i = 1, \dots, k$.

Lemma 4.1. It holds that $\mathcal{E}_\kappa \subset \mathcal{D}_{k1}$ if $\kappa \le k$, especially $\mathcal{E}_k = \mathcal{D}_{k1}$.

Proof. Without loss of generality, we assume that $\mathrm{Span}[1, s_l(\theta), s_l^{(1)}(\theta) | l = 1, \dots, \kappa] = \mathrm{Span}[1, s_l(\theta), s_m^{(1)}(\theta) | l = 1, \dots, \kappa; m = 1, \dots, r]$. Then there exists a $(\kappa + r + 1) \times (\kappa - r)$ constant matrix R such that

$$ {}^t\boldsymbol{s}^{(1)}_{\cdot\kappa-r}(\theta) = \left(1, {}^t\boldsymbol{s}_\kappa(\theta), {}^t\boldsymbol{s}^{(1)}_{r\cdot}(\theta)\right) R \tag{4.3} $$

for all $\theta \in \mathbb{R}^1$, where $\boldsymbol{s}_{r\cdot}(\theta) := {}^t(s_1(\theta), \dots, s_r(\theta))$, $\boldsymbol{s}_{\cdot\kappa-r}(\theta) := {}^t(s_{r+1}(\theta), \dots, s_\kappa(\theta))$. Since $f(x - \theta)$ is of the form (4.1), we see that

$$ \left(1, {}^t\boldsymbol{s}_\kappa(\theta), {}^t\boldsymbol{s}^{(1)}_{r\cdot}(\theta)\right) \begin{pmatrix} t_0^{(2)}(x) \\ \boldsymbol{t}_\kappa^{(2)}(x) \\ \boldsymbol{t}_{r\cdot}^{(1)}(x) \end{pmatrix} + {}^t\boldsymbol{s}^{(1)}_{\cdot\kappa-r}(\theta) \boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x) = 0, $$

where $\boldsymbol{t}_{r\cdot}(x)$ and $\boldsymbol{t}_{\cdot\kappa-r}(x)$ are defined similar to $\boldsymbol{s}_{r\cdot}(\theta)$ and $\boldsymbol{s}_{\cdot\kappa-r}(\theta)$, respectively. By (4.3) and the linear independency of $1, {}^t\boldsymbol{s}_\kappa(\theta), {}^t\boldsymbol{s}^{(1)}_{r\cdot}(\theta)$, we get $\left(t_0^{(2)}(x), {}^t\boldsymbol{t}_\kappa^{(2)}(x), {}^t\boldsymbol{t}_{r\cdot}^{(1)}(x)\right) = -{}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x){}^tR$. Put ${}^tR =: \left({}^tR_0, {}^tR_1, {}^tR_2\right)$ where R_0, R_1 and R_2 are $(r+1) \times (\kappa - r)$, $(\kappa - r) \times (\kappa - r)$ and $r \times (\kappa - r)$ matrices, respectively. Then it follows that

$$ \begin{cases} \left(t_0^{(2)}(x), {}^t\boldsymbol{t}_{r\cdot}^{(2)}(x)\right) = -{}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x){}^tR_0, \\ {}^t\boldsymbol{t}^{(2)}_{\cdot\kappa-r}(x) = -{}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x){}^tR_1, \\ {}^t\boldsymbol{t}^{(1)}_{r\cdot}(x) = -{}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x){}^tR_2. \end{cases} $$

Using these equalities, we get

$$ h_1(x - \theta) = {}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x) \left(\boldsymbol{s}_{\cdot\kappa-r}(\theta) - {}^tR_2\boldsymbol{s}_{r\cdot}(\theta)\right) + t_0^{(1)}(x). $$

It follows that

$$ h_1^{(m)}(x - \theta) = (-1)^m\, {}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x)\, {}^tR_1^m \left(\boldsymbol{s}_{\cdot\kappa-r}(\theta) - {}^tR_2\boldsymbol{s}_{r\cdot}(\theta)\right) + t_0^{(m+1)}(x) \tag{4.4} $$

for $m = 1, 2, \dots$, since

$$ {}^t\boldsymbol{t}^{(m+1)}_{\cdot\kappa-r}(x) = (-1)^m\, {}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x){}^tR_1^m. $$

Let $\varphi_{R_1}(t)$ be the characteristic polynomial of R_1, $\lambda_1, \dots, \lambda_{\kappa-r}$ be eigenvalues of R_1 and $W_m(\lambda)$ be a coefficient of t^m in $\varphi_{R_1}(t)$ for $m = 0, \dots, \kappa - r$, that is,

$$ \varphi_{R_1}(t) := \det(tE_{\kappa-r} - R_1) = \sum_{m=0}^{\kappa-r} W_m(\lambda)t^m, $$

where E_n is an n-th order unit matrix. By (4.4) we get

$$\sum_{m=0}^{\kappa-r}(-1)^m W_m(\lambda)h_1^{(m)}(x-\theta)$$

$$=\sum_{m=0}^{\kappa-r} W_m(\lambda)\,{}^t\boldsymbol{t}^{(1)}_{\cdot\kappa-r}(x){}^tR_1^m\left(\boldsymbol{s}_{\cdot\kappa-r}(\theta)-{}^tR_2\boldsymbol{s}_{r\cdot}(\theta)\right)+\sum_{m=0}^{\kappa-r}(-1)^m W_m(\lambda)t_0^{(m+1)}(x)$$

$$=\sum_{m=0}^{\kappa-r}(-1)^m W_m(\lambda)t_0^{(m+1)}(x).$$

Differentiating this equation with respect to θ, we have

$$\sum_{m=0}^{\kappa-r}(-1)^m W_m(\lambda)h_1^{(m+1)}(x-\theta)=0,$$

which shows the first assertion of Lemma 4.1.

Conversely, we should show $\mathcal{D}_{k1}\subset\mathcal{E}_k$. We suppose that there exists some $(k+1)\times k$ constant matrix $\left(\boldsymbol{a}^0_{k+1,1},\dots,\boldsymbol{a}^0_{k+1,k}\right)$ such that $(4.2)_1$ is satisfied for all $u\in\mathbb{R}^1$. Then we consider the $(k+2)$-th order algebraic equation

$$z\left|\left(\boldsymbol{z}_{k+1},\boldsymbol{a}^0_{k+1,1},\dots,\boldsymbol{a}^0_{k+1,k}\right)\right|=0, \tag{4.5}$$

since we are concerned with $\log f(u)$ instead of $h_1(u)$ (cf. (3.7)). Denote the solutions of the equation (4.5) by $z=z_\alpha$ and the algebraic multiplicities of z_α by w_α for $\alpha=0,1,\dots,\eta$, where $z_0:=0$ and $\sum_{\alpha=0}^{\eta}w_\alpha=k+2$. Then $\log f(u)$ can be represented as

$$\log f(u)=-\int h_1(u)du=-\sum_{\delta=1}^{w_0}H_\delta u^{\delta-1}-\sum_{\alpha=1}^{\eta}\sum_{\beta=1}^{w_\alpha}H_{\alpha\beta}u^{\beta-1}e^{z_\alpha u},$$

where $H_{\alpha\beta}$, H_δ ($\alpha=1,\dots,\eta;\beta=1,\dots,w_\alpha;\delta=1,\dots,w_0$) are arbitrary complex constants. Thus $\log f(x-\theta)$ is of the form

$$\log f(x-\theta)=s_0(\theta)+\sum_{\gamma_1=1}^{w_0-2}s_{\gamma_1}(\theta)t_{\gamma_1}(x)+t_0(x)+\sum_{\alpha=1}^{\eta}\sum_{\gamma_2=0}^{w_\alpha-1}s_{\alpha\gamma_2}(\theta)t_{\alpha\gamma_2}(x),$$

where

$$s_0(\theta)=-H_{w_0}(-\theta)^{w_0-1},\quad s_{\gamma_1}(\theta)=-(-\theta)^{\gamma_1},\quad s_{\alpha\gamma_2}(\theta)=-(-\theta)^{\gamma_2}e^{-z_\alpha\theta},$$

$$t_0(x)=-\sum_{\delta=1}^{w_0}H_\delta x^{\delta-1},\quad t_{\gamma_1}(x)=\sum_{\delta=\gamma_1+1}^{w_0}H_\delta\binom{\delta-1}{\gamma_1}x^{\delta-\gamma_1-1},$$

$$t_{\alpha\gamma_2}(x)=\sum_{\beta=\gamma_2+1}^{w_\alpha}H_{\alpha\beta}\binom{\beta-1}{\gamma_2}e^{z_\alpha x}x^{\beta-\gamma_2-1},$$

which implies that $\mathcal{D}_{k1} \subset \mathcal{E}_k$ since $1, s_{\gamma_1}(\theta), s_{\alpha\gamma_2}(\theta)$ $(\alpha = 1, \ldots, \eta; \gamma_1 = 1, \ldots, w_0 - 2; \gamma_2 = 0, 1, \ldots, w_\alpha - 1)$ are linearly independent. □

From Lemma 4.1, we get the next theorem which coincides with Theorem 1 in Ferguson (1962).

Theorem 3. Under the above notations, a κ-th dimensional exponential family involving one location parameter is limited to the form

$$f(x-\theta) = \exp\left[-\sum_{\delta=1}^{w_0} H_\delta (x-\theta)^{\delta-1} - \sum_{\alpha=1}^{\eta}\sum_{\beta=1}^{w_\alpha} H_{\alpha\beta}(x-\theta)^{\beta-1} e^{z_\alpha(x-\theta)}\right].$$

Remark 4.1. The setting in Lemma 4.1 is somewhat different from Ferguson (1962), who assumed the linearly independency of $1, t_1(x), \ldots, t_\kappa(x)$ instead of the smoothness of $t_1(x), \ldots, t_\kappa(x)$, that is, a full rank exponential family. If we assume the conditions of Ferguson, then the latter part of Lemma 4.1 is not right, since the functions $1, t_{\gamma_1}(x), t_{\alpha\gamma_2}(x)$ $(\alpha = 1, \ldots, \eta; \gamma_1 = 1, \ldots, w_0 - 2; \gamma_2 = 0, 1, \ldots, w_\alpha - 1)$ are not necessarily linearly independent.

Remark 4.2. By Lemma 3.1 and Lemma 4.1, we see that $\mathcal{B}_k \subset \cap_{i=1}^k \mathcal{F}_{ki} \subset \mathcal{D}_{k1} = \mathcal{E}_k$. Therefore, a family of distributions attaining the Bhattacharyya bound is included in an exponential family in a location parameter family.

5. Scale parameter family

In this section we shall specify a scale parameter family of distributions attaining the Bhattacharyya bound. Let $f(x,\theta) = f(x/\theta)/\theta$ where $x > 0$ and $\theta > 0$. We transform the variables x, θ and the functions $\hat{g}(\cdot)$, $g(\cdot)$, $f(\cdot,\cdot)$, $a_{ki}(\cdot)$ $(i = 1, \ldots, k)$ as

$$y := \log x,\ \sigma := \log\theta,\ \hat{g}^*(y) := \hat{g}(e^y),\ g^*(\sigma) := g(e^\sigma),\ f^*(u) := f(e^u),\ a^*_{ki}(\sigma) := a_{ki}(e^\sigma)$$

(see Ferguson (1962)). Using these transformations, we have $f(x,\theta) = e^{-\sigma} f^*(y-\sigma)$.

Lemma 5.1. For each $j = 0, 1, \ldots, k$, the function $(\partial^j/\partial\theta^j) f(x,\theta)$ can be represented as

$$\frac{\partial^j}{\partial\theta^j} f(x,\theta) = -\left(-e^{-\sigma}\right)^{j+1}\left(f^*(y-\sigma), f^{*(1)}(y-\sigma), \ldots, f^{*(k)}(y-\sigma)\right) \boldsymbol{b}_{k+1,j} \tag{5.1}$$

for some constant vector $\boldsymbol{b}_{k+1,j} := {}^t(b_{j,0}, \ldots, b_{j,j}, 0, \ldots, 0)$.

Proof. It is clear that (5.1) is satisfied for $j = 0$. By induction, suppose that (5.1) holds for $j = 0, 1, \ldots, j_0 < k$. Then

$$\begin{aligned}
&\frac{\partial^{j_0+1}}{\partial\theta^{j_0+1}} f(x,\theta) \\
&= -\frac{\partial}{\partial\sigma}\left\{(-e^{-\sigma})^{j_0+1}\left(f^*(y-\sigma), f^{*(1)}(y-\sigma), \ldots, f^{*(k)}(y-\sigma)\right) \boldsymbol{b}_{k+1,j_0}\right\} e^{-\sigma} \\
&= -(-e^{-\sigma})^{j_0+2}\left(f^*(y-\sigma), f^{*(1)}(y-\sigma), \ldots, f^{*(k)}(y-\sigma)\right) \boldsymbol{b}_{k+1,j_0+1},
\end{aligned}$$

where

$$\boldsymbol{b}_{k+1,j_0+1} := \begin{pmatrix} j_0+1 & & & O \\ 1 & j_0+1 & & \\ & 1 & \ddots & \\ O & & 1 & j_0+1 \end{pmatrix} \boldsymbol{b}_{k+1,j_0}$$

which implies that (5.1) holds for $j = j_0 + 1$. □

By Lemma 5.1, we get

$$\left(1, \frac{(\partial/\partial\theta) f(x,\theta)}{f(x,\theta)}, \ldots, \frac{(\partial^k/\partial\theta^k) f(x,\theta)}{f(x,\theta)}\right) = \left(1, \frac{f^{*(1)}(y-\sigma)}{f^*(y-\sigma)}, \ldots, \frac{f^{*(k)}(y-\sigma)}{f^*(y-\sigma)}\right) B_{k+1}(\sigma),$$

where

$$B_{k+1}(\sigma) := \left(\boldsymbol{b}_{k+1,0}, (-e^{-\sigma})\boldsymbol{b}_{k+1,1}, \ldots, (-e^{-\sigma})^k \boldsymbol{b}_{k+1,k}\right).$$

Thus (2.2) is equivalent to

$$\hat{g}^*(y) = {}^t\left(B_{k+1}(\sigma)\tilde{a}^*_{k+1}(\sigma)\right) \tilde{h}^*_{k+1}(y-\sigma), \tag{3.1*}$$

where $\tilde{a}^*_{k+1}(\sigma) := {}^t(a^*_{k0}(\sigma), a^*_{k1}(\sigma), \ldots, a^*_{kk}(\sigma))$ and $\tilde{h}^*_{k+1}(u) := {}^t(1, f^{*(1)}(u)/f^*(u), \ldots, f^{*(k)}(u)/f^*(u))$. Put $\mathcal{A}^*_k := \{\tilde{a}_{k+1}(\cdot) | B_{k+1}(\sigma)\tilde{a}^*_{k+1}(\sigma) \in \mathcal{A}_k\}$, and

$$\mathcal{B}^*_k := \left\{f(\cdot) | \exists\hat{g}(\cdot) \text{ and } \exists\tilde{a}_{k+1}(\cdot) \in \mathcal{A}^*_k \text{ s.t. (3.1)}^* \text{ is satisfied for all } y, \sigma \in \mathbb{R}^1\right\}.$$

In a similar way to the location parameter case, we see that if $f \in \mathcal{B}^*_k$ then $f^*(u)$ is limited to the case when

$$f^*(u) = \begin{cases} \exp\{-(H_0 + H_1 u + H_2 u^2/2)\} & \text{if the solutions of (3.7) are only } 0, \\ \exp\{-(H_0 + H_1 u + H_2 e^{z_0 u}/z_0)\} & \text{if the solutions of (3.7) are } 0, z_0, z_0^2, \ldots, z_0^k, \\ \exp\{-(H_0 + H_1 u)\} & \text{otherwise,} \end{cases}$$

where H_0, H_1 and H_2 are arbitrary complex constants and z_0 is arbitrary non-zero complex constant. So we get the next theorem.

Theorem 4. A scale parameter family of distributions attaining the Bhattacharyya lower bound consists of a log-normal distribution and an extended normal and gamma distribution.

Proof. (I) First case (log-normal). Under the condition that $f(x,\theta)$ is a p.d.f. over $(\mathcal{X},\mathcal{B})$, $f(x,\theta)$ can be represented as

$$f(x,\theta)=\frac{1}{\sqrt{2\pi}bx}\exp\left\{-\frac{1}{2b^2}\left(\log\frac{x}{\theta}-a\right)^2\right\}\quad (x>0;\theta>0;a\in\mathbb{R}^1,b>0).$$

(II) Second case (extended normal and gamma). Under the condition that $f(x,\theta)$ is a p.d.f. over $(\mathcal{X},\mathcal{B})$, $f(x,\theta)$ can be represented as

$$f(x,\theta)=\frac{|b|}{\Gamma(a)c^a\theta}\left(\frac{x}{\theta}\right)^{ab-1}\exp\left\{-\frac{1}{c}\left(\frac{x}{\theta}\right)^b\right\}\quad (x>0;\theta>0;a,c>0,b\neq 0).$$

(III) Final case (extension of a triangular). In this case, $f(x,\theta)$ has the form

$$f(x,\theta)=\frac{1}{\theta}e^{-H_0}\left(\frac{x}{\theta}\right)^{-H_1}.$$

It is impossible to consider the Bhattacharyya inequality since the definite integral of $f(x,\theta)$ over $(0,\infty)$ diverges for any H_0 and H_1.

We can justify the theorem by the suitable transformation. □

6. Concluding remarks

In this paper, we specified the family of distributions attaining the Bhattacharyya bound for a location and a scale parameter family under suitable conditions. We conclusively remark that it is possible to specify a family of distributions attaining the Bhattacharyya bound if the equation (2.2) can be reduced to the form (3.1) by a suitable transformation.

In some papers which discuss the Bhattacharyya inequality, a family of distributions is restricted to the exponential family so that $f(x,\theta)=\exp\{t(x)\psi_1(\theta)-\psi_2(\theta)\}$ where $\psi_2'(\theta)/\psi_1'(\theta)=\theta$ and $1/\psi_1'(\theta)$ is a quadratic polynomial in θ (see Blight and Rao (1974), Khan (1984), and also Shanbhag (1972)). The four types of distributions derived in this paper, i.e., a normal distribution and an exp-gamma distribution for a location parameter, and a log-normal distribution and an extended normal and gamma distribution for a scale parameter do not directly belong to the above restricted exponential family. But, by transforming the parameter appropriately, the four distributions are in that family.

Finally, the case when the determinant of the coefficient matrix $(\boldsymbol{a}_{k+1,0}(\theta),\ldots,\boldsymbol{a}_{k+1,k}(\theta))$ is zero still remains open. In that case, the author conjectures that the support of $f(x-\theta)$

does depend on the unknown parameter θ under the essential condition that the function $f(\cdot)$ is a p.d.f. over $(\mathbb{R}^1, \mathcal{B})$. This shows that it is nonsense to consider the Bhattacharyya inequality.

References

Bhattacharyya, A. (1946). On some analogues of the information and their use in statistical estimation. *Sankhyā*, **8**, 1–14.

Bondesson, L. (1975). Uniformly minimum variance estimation in location parameter families. *Ann. Statist.*, **3**, 637–660.

Blight, J. N. and Rao, P. V. (1974). The convergence of Bhattacharyya bounds. *Biometrika*, **61**, 137–142.

Dynkin, E. B. (1961). Necessary and sufficient statistics for a family of probability distributions. *Selected Translations in Mathematical Statistics and Probability.* **1**, 17–40.

Fend, A. V. (1959). On the attainment of Cramér-Rao and Bhattacharyya bounds for the variance of an estimate. *Ann. Math. Statist.*, **30**, 381–388.

Ferguson, T. S. (1962). Location and scale parameters in exponential families of distributions. *Ann. Math. Statist.*, **33**, 986–1001.

Kakeshita, S. (1962). Generalized efficient estimates and its attainable parametric functions. *Bull. Math. Statist.*, **10**, 5–15.

Khan, R. A. (1984). On UMVU estimators and Bhattacharyya bounds in exponential distributions. *J. Statist. Plann. Inference*, **9**, 199–206.

Mase, S. (1977). On the attainment of the supremum of the Bhattacharyya lower bounds for variances of estimators. *Austral. J. Statist.*, **19**, 144–151.

Shanbhag, D. N. (1972). Some characterizations based on the Bhattacharya matrix. *J. Appl. Probability*, **9**, 580–587.

Takeuchi, K. (1973). On location parameter family of distributions with uniformly minimum variance unbiased estimator of location. *Proceedings of the Second Japan-USSR Symposium on Probability Theory.* Lecture Notes in Math., **330**, 465–477. Springer, Berlin.

Tanaka, H. (2003). On a relation between a family of distributions attaining the Bhattacharyya bound and that of linear combinations of the distributions from an exponential family. To appear in *Comm. Statist. Theory and Method.*, **32**, 1885–1896.

Tanaka, H. and Akahira, M. (2003). On a family of distributions attaining the Bhattacharyya bound. *Ann. Inst. Statist. Math.*, **55**, 309–317.

Wijsman, R. A. (1973). On the attainment of the Cramér-Rao lower bound. *Ann. Statist.*, **1**, 538–542.

Zacks, S. (1971). *The Theory of Statistical Inference.* Wiley, New York.

Institute of Mathematics
University of Tsukuba
Ibaraki 305-8571
Japan
E-mail:hide@math.tsukuba.ac.jp